Algebra 2/Pre-Calculus

Name_____

Applications of the Pythagorean Identities (Trigonometry, Day 7)

In this handout, we will apply our knowledge of the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$, to equation solving. We will also see how we can use this identity to generate other trigonometric identities. Unless otherwise stated, all of the problems on this handout should be done **without the aid of a calculator**.

1. Prove the identity $\cos^2\theta + \sin^2\theta = 1$. Write your proof clearly and include a diagram.

2. Determine whether each of the following statements is true or false. ***Note:*** Answers are provided at the end of this problem.

a. $1 - \cos^2\theta = \sin^2\theta$

b. $\sin^2\theta - 1 = \cos^2\theta$

c. $1-\sin^2\theta=\cos^2\theta$

d. $\cos^2\theta-1=-\sin^2\theta$

e. $(1-\cos\theta)(1+\cos\theta)=1-\cos^2\theta$

f. $(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)=1$

g. $1-\sin^2\theta=(1+\sin\theta)(1-\sin\theta)$

h. $\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$

i. $\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$

Answers

a. True b. False c. True d. True e. True f. False g. True h. True i. True

3. In this problem, we will combine our knowledge of the Pythagorean identity with our knowledge of other trigonometric identities.

a. Simplify $\sin(90^\circ - \theta)$.

b. You should have found that $\sin(90^\circ - \theta) = \cos\theta$. Now simplify $\sin^2\theta + \sin^2(90^\circ - \theta)$.

c. Since $\sin(90^\circ - \theta) = \cos\theta$, we can make the following conclusion:

$$\sin^2\theta + \sin^2(90^\circ - \theta) = \sin^2\theta + \cos^2\theta = 1$$

Now simplify $1 - \sin^2(180^\circ - \theta)$.

d. Since $\sin(180^\circ - \theta) = \sin\theta$, we can conclude that $1 - \sin^2(180^\circ - \theta) = 1 - \sin^2\theta = \cos^2\theta$.

Now simplify $(1 + \sin\theta)(1 + \sin(-\theta))$. ***Note:*** Answers are provided at the end of this problem.

e. Simplify $(1 + \cos\theta)(1 + \cos(180^\circ - \theta))$.

f. Simplify $\sin^2(\theta + 90^\circ) - 1$.

Some answers d. $\cos^2\theta$ e. $\sin^2\theta$ f. $-\sin^2\theta$

4. The goal of the next few problems is to explore the connections between trigonometric equations and polynomials, especially quadratics. We begin by reviewing factoring.

Factor each of the following.

a. $x^2 - 1$

b. $2x^2 + x$

c. $1 - x^2$

d. $2x^2 + 3x + 1$

e. $2x^2 + 14x - 36$

f. $6x^2 + x - 12$

g. $2x^3 - x^2 - x$

h. $4x^3 - 4x^2 + x$

i. $x^4 - 8x^2 + 15$

j. $x^3 + 7x^2 - 6x - 42$

Answers a. $(x+1)(x-1)$ b. $x(2x+1)$ c. $(1+x)(1-x)$ d. $(2x+1)(x+1)$

e. $2(x+9)(x-2)$ f. $(3x-4)(2x+3)$ g. $x(2x+1)(x-1)$ h. $x(2x-1)^2$

i. $(x^2-5)(x^2-3)$ j. $(x^2-6)(x+7) = (x+\sqrt{6})(x-\sqrt{6})(x+7)$

5. We can apply the same ideas about factoring to expressions that involve trigonometric functions. Factor each of the following. ***Note:*** Answers are provided at the end of this problem.

a. $1-\cos^2\theta$

b. $3\sin^2\theta+\sin\theta$

c. $\cos^2\theta-7\cos\theta+12$

d. $2\sin^2\theta+\sin\theta-1$

e. $\cos^3\theta+\cos^2\theta-20\cos\theta$

f. $4\cos^2\theta+7\cos\theta+3$

Answers a. $(1+\cos\theta)(1-\cos\theta)$ b. $\sin\theta(3\sin\theta+1)$ c. $(\cos\theta-4)(\cos\theta-3)$

d. $(2\sin\theta-1)(\sin\theta+1)$ e. $\cos\theta(\cos\theta+5)(\cos\theta-4)$ f. $(4\cos\theta+3)(\cos\theta+1)$

6. In this problem, we will explore another technique we can use when factoring with trigonometric functions.

a. Factor $\cos^2\theta + \sin\theta - 1$. ***Note:*** If you need a hint, look ahead to part **b**. But try it on your own first.

b. Remember that $\cos^2\theta + \sin^2\theta = 1$. How does this fact help us to factor $\cos^2\theta + \sin\theta - 1$? (Factor it, if you can.)

c. Since $\cos^2\theta + \sin^2\theta = 1$, we know that $\cos^2\theta = 1 - \sin^2\theta$. Hence, we can factor $\cos^2\theta + \sin\theta - 1$ in the following way:

$$\cos^2\theta + \sin\theta - 1 = 1 - \sin^2\theta + \sin\theta - 1 = \sin\theta - \sin^2\theta = \sin\theta(1 - \sin\theta)$$

Now use a similar approach to factor $1 - 2\sin^2\theta - \cos\theta$.

Answers a. $\sin\theta(1-\sin\theta)$ b. $\sin\theta(1-\sin\theta)$ c. $(2\cos\theta+1)(\cos\theta-1)$

7. Solve each of the following polynomial equations. ***Note:*** Answers are provided at the end of this problem.

a. $x^2 = 5x + 14$

b. $x^2 - 1 = 0$

c. $3x^2 - 5x - 2 = 0$

d. $x^3 + 35x = 12x^2$

e. $x^2 = \frac{1}{2}$

f. $2x^3 = x^2 + x$

Answers a. 7, -2 b. 1, -1 c. $-\frac{1}{3}$, 2 d. 0, 5, 7 e. $\pm\frac{\sqrt{2}}{2}$ f. 0, $-\frac{1}{2}$, 1

8. Suppose we are trying to solve the equation $2\sin^2\theta = \sin\theta + 1$ for $0^\circ \le \theta < 360^\circ$.

a. Solve $2\sin^2\theta = \sin\theta + 1$ for $0^\circ \le \theta < 360^\circ$. ***Note:*** There is a hint in part **b**. But try it on your own first.

b. Here's the beginning of the solution:

$$2\sin^2\theta = \sin\theta + 1$$

$$2\sin^2\theta - \sin\theta - 1 = 0$$

$$(2\sin\theta + 1)(\sin\theta - 1) = 0$$

Now finish solving for θ.

c. Here's the full solution to the last problem:

$$2\sin^2\theta = \sin\theta + 1$$

$$2\sin^2\theta - \sin\theta - 1 = 0$$

$$(2\sin\theta + 1)(\sin\theta - 1) = 0$$

$$2\sin\theta + 1 = 0 \text{ or } \sin\theta - 1 = 0$$

$$\sin\theta = -\tfrac{1}{2} \text{ or } \sin\theta = 1$$

$$\theta = 210^\circ \text{ or } \theta = 330^\circ \text{ or } \theta = 90^\circ$$

Now solve the equation $2\cos^2\theta = \cos\theta$ for $0^\circ \le \theta < 360^\circ$.

Answer a. $\theta = 210^\circ$ or $\theta = 330^\circ$ or $\theta = 90^\circ$ b. $\theta = 210^\circ$ or $\theta = 330^\circ$ or $\theta = 90^\circ$
c. $\theta = 60^\circ$ or $\theta = 90^\circ$ or $\theta = 270^\circ$ or $\theta = 300^\circ$

9. We can apply the same ideas about equation solving to equations that involve trigonometric functions. Solve each of the following for $0^\circ \le \theta < 360^\circ$. ***Note:*** Answers are provided at the end of this problem.

a. $\sin^2\theta - \sin\theta = 0$ ***Hint:*** Start by factoring out $\sin\theta$.

b. $2\sin^2\theta = \sin\theta$ ***Hint:*** Start by moving all the terms to one side. Then factor and solve.

c. $2\cos^2\theta + \cos\theta = 0$

d. $\cos^2\theta = -\cos\theta$

e. $2\sin^2\theta - 1 = \sin\theta$

f. $4\cos^2\theta = 4\cos\theta - 1$

g. $1 = \sin^2\theta + \cos\theta$

h. $3 = 2\cos^2\theta + 3\sin\theta$

Answers a. 0, 90, 180 b. 0, 30, 150, 180 c. 90, 270, 120, 240 d. 90, 270, 180
e. 90, 210, 330 f. 60, 300 g. 0, 90, 270 h. 30, 90, 150