Algebra 2/Pre-Calculus **Name__________________**

More Problems (Day 6, Pascal's Triangle)

The goal of this handout is to explore the connections between Pascal's Triangle and certain probability problems.

1. Suppose a pizzeria offers six toppings: Pepperoni (P), Sausage (S), Onions (O), Mushrooms (M), Chicken (C), and Broccoli (B).

a. How many 2 topping pizzas are possible?

b. How many 3 topping pizzas are possible?

c. How many 1 topping pizzas are possible?

d. How many 0 topping pizzas are possible? ***Hint:*** What is a "0 topping pizza?"

e. When we defined factorials, we said that $0! = 1$. How does this relate to the question in part **d**?

f. How many total pizzas does this pizzeria offer? ***Hint:*** There are two possibilities for each topping: On the pizza or not on the pizza.

g. Consider the following identity:

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}=2^6$$

Explain this relationship in the context of the pizzeria.

h. Find the following sum: $1+7+21+35+35+21+7+1$. How does the answer relate to the last problem?

i. How many 2 topping pizzas are possible? How many 4 topping pizzas are possible?

j. You should have found that the number of 2 topping pizzas was equal to the number of 4 topping pizzas explain why this makes sense.

Answers a. $\binom{6}{2} = 15$ b. $\binom{6}{3} = 20$ c. $\binom{6}{1} = 6$

d. $\binom{6}{0} = 1$ There's only one 0 topping pizza: plain cheese.

e. We need to define $0! = 1$ so that $\binom{6}{0} = 1$. Remember, $\binom{6}{0} = \frac{6!}{0!(6-0)!}$

f. $2^6 = 64$ g. The total number of pizzas is the number of 0 topping pizzas plus the number of 1 topping pizzas plus the number of 2 topping pizzas, etc.

h. $1+7+21+35+35+21+7+1 = 2^7$ The numbers on the 7^{th} row of Pascal's Triangle sum to 2^7. i. 15 for both j. Choosing 2 toppings to be on the pizza is the same as choosing 2 topping to be off the pizza.

2. Evaluate the sum: $\sum_{k=0}^{8}\binom{8}{k}$

Answer 256

3. Find the value(s) of m and n that satisfy each equation.

a. $\binom{10}{5}=\binom{9}{5}+\binom{9}{m}$ **b.** $\binom{15}{6}=\binom{14}{6}+\binom{14}{m}$

Answer a. $m=4$ or $m=5$ b. $m=5$ or $m=9$

4. How many entries are in the nth row of Pascal's Triangle?

Answer There are $n+1$ entries.

5. What is the value of $\binom{15}{9}$ in Pascal's Triangle? ***Hint:*** Use the formula for combination numbers.

Answer $\binom{15}{9}=\frac{15!}{9!(15-9)!}=5{,}005$

6. Use the Binomial Theorem to expand each binomial. Hint: For both problems, start by finding $(a+b)^8$.

a. $(2x+y)^8$

b. $(2x-3y)^8$

Answers

a. $256x^8+1024x^7y+1792x^6y^2+1792x^5y^3+1120x^4y^4+448x^3y^5+112x^2y^6+16xy^7+y^8$

b. $256x^8-3072x^7y+16{,}128x^6y^2-48{,}384x^5y^3$
$+90{,}720x^4y^4-108{,}864x^3y^5+81{,}648x^2y^6-34{,}992xy^7+6561y^8$

7. Consider the expansion of $(a+b)^9$.

a. What is the coefficient of the a^4b^5 term? Can you find this by using combination numbers?

b. What other term or terms share this coefficient?

c. Which terms of this expansion do not share coefficients with any other terms? Why?

d. Now consider the expansion of $(a+b)^{10}$. Which coefficient of this expansion is not repeated?

Answers a. ${}_9C_4 = 126$ b. a^5b^4 c. None. There are 10 terms in the 9th row of Pascal's triangle, so every coefficient is repeated exactly twice. d. ${}_{10}C_5 = 252$

8. Consider the expansion of $(a+b)^{24}$, but don't actually do it out.

a. What is the coefficient of the a^3b^{21} term? Find this by using combination numbers, not by finding the 24th row of Pascal's Triangle.

b. Find the coefficient of the a^7b^{17} term.

c. What is the term with the largest coefficient? What is that coefficient?

Answers a. 2,024 b. 346,104 c. ${}_{24}C_{12} = 2{,}704{,}156$

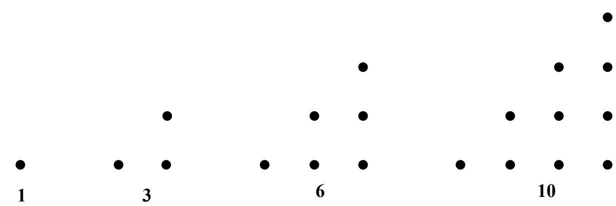
9. Now consider the expansion of $(2x+y)^{24}$ (but don't actually do it out).

a. What is the coefficient of the x^3y^{21} term? ***Hint:*** Think about the expansion for $(a+b)^{24}$. What do you need to plug in for *a* and *b*?

b. Find the coefficient of the x^6y^{18} term.

Answers a. $\binom{24}{3}\cdot 2^3 = 16{,}192$ b. $\binom{24}{6}\cdot 2^6 = 8{,}614{,}144$

10. Recall the triangular numbers that we introduced earlier in this course. The first four triangular numbers are pictured below.



a. We say write $t(1)=1$, $t(2)=3$, $t(3)=6$, and $t(4)=10$. Find the values for the next four triangular numbers.

b. Complete the following formula: $t(n)=\sum_{k=1}^{n}(???)$. (Fill in the question marks.)

c. Find an explicit formula for $t(n)$. Prove it, if you can.

d. Create a Pascal's Triangle. Include rows 0 through 8. ***Important:*** Do **neat**, **clear**, **well organized** work! If you naturally have large handwriting, consider writing smaller.

e. Do the triangular numbers appear in Pascal's Triangle? Explain.

f. Write a formula relating the triangular numbers to the combination numbers.

g. Suppose $T(n) = \sum_{k=1}^{n} t(k)$. Find the values of $T(1)$, $T(2)$, $T(3)$, $T(4)$.

h. How are the numbers of $T(1)$, $T(2)$, $T(3)$, $T(4)$, etc. related to Pascal's Triangle? Explain.

i. Write a formula relating the values of $T(n)$ to the combination numbers.

j. Now consider the values for $\sum_{k=1}^{n} T(k)$. How are these related to Pascal's Triangle? Explain.

Some answers a. $t(5)=15$, $t(6)=21$, $t(7)=28$, $t(8)=36$ b. $t(n)=\sum_{k=1}^{n} k$

c. $t(n)=\frac{n(n+1)}{2}$ e. The triangular numbers all appear along a "diagonal" of Pascal's Triangle: $t(1)={}_2C_2$, $t(2)={}_3C_2$, $t(3)={}_4C_2$, $t(4)={}_5C_2$, etc. They also form a "diagonal" on the opposite side of Pascal's Triangle: $t(1)={}_2C_0$, $t(2)={}_3C_1$, $t(3)={}_4C_2$, $t(4)={}_5C_3$, etc. f. There are two ways to write the formula: $t(n)={}_{n+1}C_2$ or $t(n)={}_{n+1}C_{n-1}$ g. $T(1)=1$, $T(2)=4$, $T(3)=10$, $T(4)=20$ h. These also form a diagonal of Pascal's Triangle: $T(1)={}_3C_3$, $T(2)={}_4C_3$, $T(3)={}_5C_3$, $T(4)={}_6C_3$, etc. i. $T(n)={}_{n+2}C_3$ j. These are the next diagonal in Pascal's Triangle: 1, 5, 15, 35, 70, etc. In general, $\sum_{k=1}^{n} T(k)={}_{n+2}C_4$.

11. Optional Challenge How many odd numbers are there on the 10th row of Pascal's Triangle? On the 20th row of Pascal's Triangle? On the 100th row? ***Note:*** This problem is hard! Start by making a table and look for patterns.